

LINES OF CURVATURE ON QUADRIC HYPERSURFACES OF \mathbb{R}^4

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ABSTRACT. Here are described the geometric structures of the lines of principal curvature and the partially umbilic singularities of the tridimensional non compact generic quadric hypersurfaces of \mathbb{R}^4 . This includes the *ellipsoidal hyperboloids* of one and two sheets and the *toroidal hyperboloids*. The present study complements the analysis of the compact *ellipsoidal hypersurfaces* carried out in [9].

1. INTRODUCTION

The first example of a *principal curvature configuration* on a surface in \mathbb{R}^3 , consisting of the umbilic points (at which the principal curvatures coincide) and, outside them, by the foliations with the minimal and maximal principal curvature lines was determined for the case of the ellipsoid $q(x, y, z) = x^2/a^2 + y^2/b^2 + z^2/c^2 = 1$, $a > b > c > 0$. See Monge [11] and the illustration in figure 3 of this configuration deduced from Dupin theorem [13], [14].

An extension of the principal configuration for the case of ellipsoidal quadric hypersurfaces in \mathbb{R}^4 was achieved in [9]. This was preceded by work of Garcia [3] where the generic properties of principal configurations on smooth hypersurfaces were established. There was determined the principal configuration on the ellipsoid

• $Q_0 = x^2/a^2 + y^2/b^2 + t^2/c^2 + w^2/d^2 = 1$, $a > b > c > d > 0$, in \mathbb{R}^4 . It was proved that it carries four closed regular curves of *partially umbilic* points, along which two of the three principal curvatures coincide and whose transversal structures are of the Darbouxian type D_1 , as at the umbilic

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points in the ellipsoid with 3 different axes in \mathbb{R}^3 . See Fig. 3. A different proof of this result was given in [9], where a complete description of the principal configurations on all tridimensional ellipsoids, including those with some coincidence in the semi-axes a, b, c, d and exhibiting isolated umbilic points.

In this paper will be determined the principal configurations for the generic non compact quadric hypersurfaces in \mathbb{R}^4 , extending in one dimension the classical results for the hyperboloids in \mathbb{R}^3 illustrated in figures 3, 4 and 5.

By a generic non-compact quadric hypersurface in \mathbb{R}^4 is meant the unit level hypersurface defined (implicitly) by one of the non-degenerate quadratic forms Q_1 , Q_2 and Q_3 which, after an orthonormal diagonalization and scalar multiplication, can be written as follows:

- $Q_1 = x^2/a^2 + y^2/b^2 + z^2/c^2 - t^2/d^2 = 1$, $a > b > c > 0$, $d > 0$. Ellipsoidal Hyperboloid with one sheet, presented in section 5;
- $Q_2 = x^2/a^2 + y^2/b^2 - z^2/c^2 - t^2/d^2 = 1$, $a > b > 0$, $d > c > 0$. Toroidal Hyperboloid, developed in section 6;
- $Q_3 = x^2/a^2 - y^2/b^2 - z^2/c^2 - t^2/d^2 = 1$ $a > 0$, $b > c > d > 0$. Ellipsoidal Hyperboloid with two sheets, presented in section 7.

The positive orientation will be made explicit in each specific case.

This paper is organized as follows: Section 2 reviews standard definitions pertinent to principal configurations on hypersurfaces in \mathbb{R}^4 . It also explains the convention, in colors and print patterns, used in their illustrations.

Section 3 includes a review of the main results of [9] pertinent to the ellipsoid Q_0 which will be helpful for the present paper.

Section 4 complements the the classical knowledge about the principal structures on quadric surfaces in \mathbb{R}^3 with results not found in standard sources [14] and [13].

In Sections 5, 6 and 7 are described the principal configurations of generic quadrics Q_1 , Q_2 and Q_3 . The last section 8 points out to the novelty of the principal structure of transversal crossing of partially umbilic separatrix surfaces established in this work.

2. PRELIMINARIES ON PRINCIPAL CONFIGURATIONS AND COLOR CONVENTIONS

The *principal configuration* on an oriented hypersurface in \mathbb{R}^4 , with euclidean scalar product $\langle \cdot, \cdot \rangle$, consists on the umbilic points, at which the 3 principal curvatures coincide, the partially umbilic points, at which only 2 principal curvatures are equal, and the integral foliations of the three principal line fields on the complement of these sets of points, which will be referred to as the *principal regular part* of the hypersurface.

Recall that the principal curvatures $k_1 \leq k_2 \leq k_3$ are the eigenvalues of the automorphism $D(-N)$ of the tangent bundle of the hypersurface, taking N as the unit normal which defines the positive orientation.

Thus the set of partially umbilic points is the union of \mathcal{P}_{12} , where $k_1 = k_2 < k_3$, and of \mathcal{P}_{23} , where $k_3 = k_2 > k_1$. The set \mathcal{U} of umbilic points is defined by $k_1 = k_2 = k_3$.


The eigenspaces corresponding to the eigenvalues k_i will be denoted by $\mathcal{L}_i, i = 1, 2, 3$. They are line fields, well defined and, for quadrics also analytic on the *principal regular part* of the hypersurface. In fact \mathcal{L}_1 is defined and analytic on the complement of $\mathcal{U} \cup \mathcal{P}_{12}$, \mathcal{L}_3 is defined and analytic on the complement of $\mathcal{U} \cup \mathcal{P}_{23}$ and \mathcal{L}_2 is defined and analytic on the complement of $\mathcal{U} \cup \mathcal{P}_{12} \cup \mathcal{P}_{23}$.


A change in the orientation of the hypersurface produces a change of sign in the principal curvatures and an exchange of \mathcal{L}_1 and \mathcal{P}_{12} into \mathcal{L}_3 and \mathcal{P}_{23} and viceversa. The line fields \mathcal{L}_2 is preserved.

A *principal chart* (u_1, u_2, u_3) is one for which the principal line fields \mathcal{L}_i are spanned by $\partial/\partial u_i, i = 1, 2, 3$. Often they will appear as *parametrizations* φ with $\partial N/\partial u_i = -k_i \partial \varphi/\partial u_i, i = 1, 2, 3$. Principal charts are also characterized by the fact that the first and second fundamental forms $g_{ij} = \langle \partial \varphi/\partial u_i, \partial \varphi/\partial u_j \rangle$ and $b_{ij} = \langle \partial^2 \varphi/\partial u_i \partial u_j, N \rangle$ are simultaneously diagonalized. That is on principal charts $g_{ij} = b_{ij} = 0, i \neq j$, and $k_i = b_{ii}/g_{ii}, i = 1, 2, 3$

2.1. Color and Print Conventions for Illustrations in this Paper. The color convention in figures illustrating principal configurations on quadrics in \mathbb{R}^4 is as follows.

- Black (— . — . — . —) integral curves of line field \mathcal{L}_1 ,
- Blue (— — — — —) integral curves of line field \mathcal{L}_3 ,
- Red (—————): integral curves of line field \mathcal{L}_2 ,

Green (

Light Blue (

The following dictionary has been adopted for illustrations of integral curves appearing in figures 1, 2, 4, 5, 8, 9 of this paper when printed in black and white: dashed, for blue; dotted-dashed-dotted, for black; full trace, for red.

3. ELLIPSOID WITH FOUR DIFFERENT AXES IN \mathbb{R}^4 [9]

Lemma 1. *The ellipsoid*

$$(1) \quad Q_0 = \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} + \frac{t^2}{d^2} - 1 = 0, \quad a > b > c > d > 0,$$

has sixteen principal charts $(u, v, w) = \varphi_i(x, y, z, t)$, $0 > u > v > w$, where $\varphi_i^{-1} : (-c^2, -d^2) \times (-b^2, -c^2) \times (-a^2, -b^2) \rightarrow \{(x, y, z, t) : xyzt \neq 0\} \cap Q_0^{-1}(0)$ is defined by equations (2).

$$(2) \quad \begin{aligned} x^2 &= \frac{a^2(a^2 + u)(a^2 + v)(a^2 + w)}{(a^2 - b^2)(a^2 - c^2)(a^2 - d^2)}, \quad y^2 = \frac{b^2(b^2 + u)(b^2 + v)(b^2 + w)}{(b^2 - a^2)(b^2 - c^2)(b^2 - d^2)}, \\ z^2 &= \frac{c^2(c^2 + u)(c^2 + v)(c^2 + w)}{(c^2 - a^2)(c^2 - b^2)(c^2 - d^2)}, \quad w^2 = \frac{d^2(d^2 + u)(d^2 + v)(d^2 + w)}{(d^2 - a^2)(d^2 - b^2)(d^2 - c^2)}. \end{aligned}$$

For all $p \in \{(x, y, z, t) : xyzt \neq 0\} \cap Q_0^{-1}(0)$ and $Q_0 = Q_0^{-1}(0)$ positively oriented by the inward normal directed by $-\nabla Q_0$, the principal curvatures satisfy $0 < k_1(p) < k_2(p) < k_3(p)$.

The hyperplanes $x = 0$ and $z = 0$ are invariant by the foliations \mathcal{F}_1 and \mathcal{F}_2 , while the hyperplanes $y = 0$ and $t = 0$ are invariant by the foliations \mathcal{F}_2 and \mathcal{F}_3 .

The first fundamental form is given by:

$$(3) \quad \begin{aligned} I &= \frac{1}{4} \frac{(u-w)(u-v)u}{\xi_0(u)} du^2 + \frac{1}{4} \frac{(v-w)(u-v)(-v)}{\xi_0(v)} dv^2 \\ &\quad + \frac{1}{4} \frac{(v-w)(u-w)w}{\xi_0(w)} dw^2, \\ \xi_0(x) &= (d^2 + x)(c^2 + x)(b^2 + x)(a^2 + x). \end{aligned}$$

The second fundamental form is given by:

$$(4) \quad \begin{aligned} II &= -\frac{1}{4} \frac{bcda(u-w)(u-v)}{\sqrt{-uvw}\xi_0(u)} du^2 + \frac{1}{4} \frac{dcba(v-w)(u-v)}{\sqrt{-uvw}\xi_0(v)} dv^2 \\ &\quad - \frac{1}{4} \frac{(u-w)(v-w)abcd}{\sqrt{-uvw}\xi_0(w)} dw^2. \end{aligned}$$

Therefore,

$$(5) \quad k_1 = -\frac{abcd}{u\sqrt{-uvw}}, \quad k_2 = -\frac{abcd}{v\sqrt{-uvw}}, \quad k_3 = -\frac{abcd}{w\sqrt{-uvw}}.$$

A theorem proved in [9] is reviewed now.

Theorem 1. [9] *The umbilic set of the ellipsoid Q_0 in equation (1) is empty and its partially umbilic set consists of four closed curves \mathcal{P}_{12}^1 , \mathcal{P}_{12}^2 , \mathcal{P}_{23}^1 and \mathcal{P}_{23}^2 , which in the chart (u, v, w) defined by*

$$\alpha_{\pm}(u, v, w) = (au, bv, cw, \pm d\sqrt{1 - u^2 - v^2 - w^2})$$

are given by:

$$(6) \quad \begin{aligned} w=0, \quad & \frac{u^2(a^2 - d^2)}{a^2 - c^2} + \frac{v^2(b^2 - d^2)}{b^2 - c^2} = 1, \quad \text{ellipse,} \\ v=0, \quad & \frac{u^2(a^2 - d^2)}{a^2 - b^2} - \frac{w^2(c^2 - d^2)}{b^2 - c^2} = 1, \quad \text{hyperbole.} \end{aligned}$$

i) *The principal foliation \mathcal{F}_1 is singular on $\mathcal{P}_{12}^1 \cup \mathcal{P}_{12}^2$, \mathcal{F}_3 is singular on $\mathcal{P}_{23}^1 \cup \mathcal{P}_{23}^2$ and \mathcal{F}_2 is singular on $\mathcal{P}_{12}^1 \cup \mathcal{P}_{12}^2 \cup \mathcal{P}_{23}^1 \cup \mathcal{P}_{23}^2$.*

The partially umbilic curves $\mathcal{P}_{12}^1 \cup \mathcal{P}_{12}^2$ whose transversal structures are of type D_1 , have their partially umbilic separatrix surfaces spanning a cylinder C_{12} such that $\partial C_{12} = \mathcal{P}_{12}^1 \cup \mathcal{P}_{12}^2$.

Also, the partially umbilic curves \mathcal{P}_{23}^1 and \mathcal{P}_{23}^2 are of type D_1 , their partially umbilic separatrix surfaces span a cylinder C_{23} so that $\partial C_{23} = \mathcal{P}_{23}^1 \cup \mathcal{P}_{23}^2$.

All the leaves of \mathcal{F}_1 outside the cylinder C_{12} are diffeomorphic to \mathbb{S}^1 . Analogously for the principal foliation \mathcal{F}_3 outside the cylinder C_{23} . See figure 1.

ii) *The principal foliation \mathcal{F}_2 is singular at $\mathcal{P}_{12}^1 \cup \mathcal{P}_{12}^2 \cup \mathcal{P}_{23}^1 \cup \mathcal{P}_{23}^2$ and there are Hopf bands (cylinders with boundaries consisting on two linked closed curves) H_{123}^1 and H_{123}^2 such that $\partial H_{123}^1 = \mathcal{P}_{12}^1 \cup \mathcal{P}_{23}^1$ and $\partial H_{123}^2 = \mathcal{P}_{12}^2 \cup \mathcal{P}_{23}^2$, which are partially umbilic separatrix surfaces.*

All the leaves of \mathcal{F}_2 , outside the partially umbilic separatrix surfaces, are diffeomorphic to \mathbb{S}^1 . See figure 2.

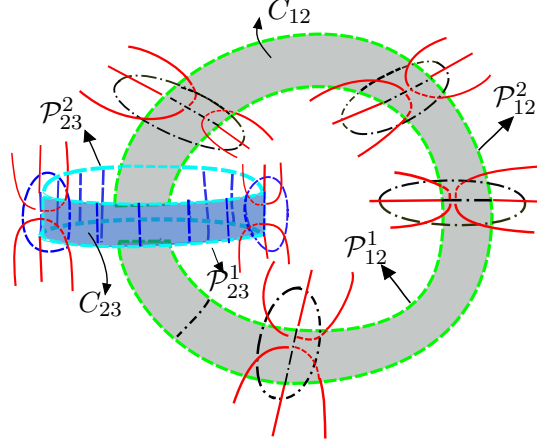


FIGURE 1. Global behavior of the principal foliations \mathcal{F}_i , ($i = 1, 2, 3$). The cylinder C_{12} is foliated by principal lines of \mathcal{F}_1 and is bounded by two partially umbilic lines (dashed green). The cylinder C_{23} is foliated by principal lines of \mathcal{F}_3 and is bounded by two partially umbilic lines (dashed light blue).

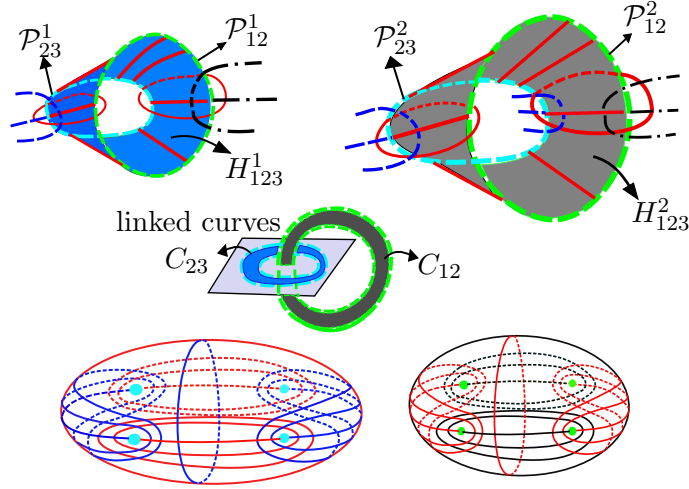


FIGURE 2. Hopf bands H^1_{123} and H^2_{123} with leaves of \mathcal{F}_2 . Global behavior of the Principal Foliations \mathcal{F}_i near the partially umbilic curves, top. Bottom: quartic ellipsoidal surface whose 4 umbilics slide along the partially umbilic closed lines (horizontal, blue dotted print, and vertical, green dotted print).

4. COMPLEMENTS ON PRINCIPAL STRUCTURES ON QUADRICS IN \mathbb{R}^3

In this section a theorem of Dupin is revisited. No presentation as explicit as the one provided here has been found in the literature. See [13, 14].

Proposition 1. *The ellipsoid*

$$(7) \quad q_0(x, y, z) = \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} - 1 = 0, \quad a > b > c > 0,$$

is diffeomorphic to \mathbb{S}^2 and has eight principal charts $(u, v) = \varphi_i(x, y, z)$, $u > v$, where $\varphi_i^{-1} : (-a^2, -b^2) \times (-b^2, -c^2) \rightarrow \{(x, y, z) : xyz \neq 0\} \cap q_0^{-1}(0)$ is defined by equations (8).

$$(8) \quad \begin{aligned} x^2 &= \frac{a^2 (a^2 + v) (a^2 + u)}{(a^2 - c^2) (a^2 - b^2)}, \quad y^2 = -\frac{b^2 (b^2 + v) (b^2 + u)}{(a^2 - b^2) (b^2 - c^2)} \\ z^2 &= -\frac{c^2 (c^2 + v) (c^2 + u)}{(a^2 - c^2) (b^2 - c^2)} \end{aligned}$$

For all $p \in \{(x, y, z) : xyz \neq 0\} \cap q_0^{-1}(0)$ and $q_0 = q_0^{-1}(0)$ positively oriented by the inward normal, the principal curvatures satisfy $k_1(p) \leq k_2(p)$ and in the chart (u, v) are given by: $k_1(u, v) = -\frac{abc}{u\sqrt{uv}}$, $k_2(u, v) = -\frac{abc}{v\sqrt{uv}}$.

The umbilic set is given by

$$\mathcal{U} = \{p : k_1(p) = k_2(p)\} = \left(\pm \sqrt{\frac{a^2 (a^2 - b^2)}{a^2 - c^2}}, 0, \pm \sqrt{\frac{c^2 (b^2 - c^2)}{a^2 - c^2}} \right).$$

The leaves of the principal foliations \mathcal{F}_1 and \mathcal{F}_2 are all closed (with the exception of the four umbilic separatrix arcs). See figure 3.

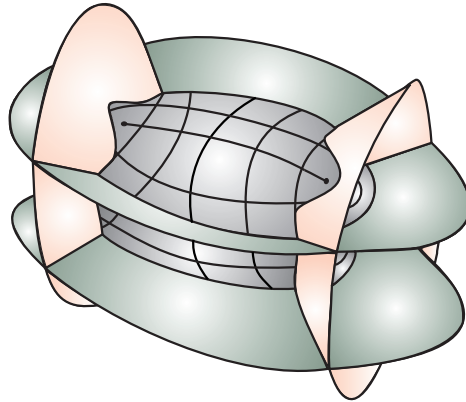


FIGURE 3. Illustration of Dupin Theorem. Principal lines of the ellipsoid Q_0 and orthogonal family of quadrics in \mathbb{R}^3 .

The proof follows from lemma 2.

Lemma 2. *Consider the ellipsoid q_0 given in \mathbb{R}^3 by equation (7) with $a > b > c > 0$. Let*

$$s_1 = \frac{1}{2} \int_{-b^2}^{-c^2} \sqrt{\frac{u}{(a^2+u)(c^2+u)}} du < \infty \quad s_2 = \frac{1}{2} \int_{-a^2}^{-b^2} \sqrt{\frac{v}{(a^2+v)(c^2+v)}} dv < \infty.$$

There exists a parametrization $\varphi : [-s_1, s_1] \times [-s_2, s_2] \rightarrow q_0 \cap \{(x, y, z), y \geq 0\}$ such that the principal lines are the coordinate curves and φ is conformal in the interior of the rectangle.

Moreover $\varphi(s_1, s_2) = U_1$, $\varphi(-s_1, s_2) = U_2$, $\varphi(-s_1, -s_2) = U_3$ and $\varphi(s_1, -s_2) = U_4$.

By symmetry considerations the same result holds for the region $q_0 \cap \{(x, y, z), y \leq 0\}$.

Proof. The ellipsoid q_0 has a principal chart (u, v) defined by the parametrization $\psi : [-b^2, -c^2] \times [-a^2, -b^2] \rightarrow \{(x, y, z) : x > 0, y > 0, z > 0\}$ given by:

$$\psi(u, v) = \left(a \sqrt{\frac{(a^2+v)(a^2+u)}{(a^2-c^2)(a^2-b^2)}}, b \sqrt{\frac{(b^2+v)(b^2+u)}{(a^2-b^2)(c^2-b^2)}}, c \sqrt{\frac{(c^2+v)(c^2+u)}{(b^2-c^2)(a^2-c^2)}} \right).$$

The fundamental forms in this chart are given by:

$$\begin{aligned} I &= Edu^2 + Gdv^2 = \frac{1}{4} \frac{u(u-v)}{h(u)} du^2 - \frac{1}{4} \frac{v(u-v)}{h(v)} dv^2, \\ II &= edu^2 + gdv^2 = \frac{abc(u-v)}{4\sqrt{uv}h(u)} du^2 - \frac{abc(u-v)}{4\sqrt{uv}h(v)} dv^2 \\ h(t) &= (a^2+t)(b^2+t)(c^2+t). \end{aligned}$$

The principal curvatures are given by $k_2(u, v) = -\frac{abc}{v\sqrt{uv}}$, $k_1(u, v) = -\frac{abc}{u\sqrt{uv}}$. Therefore, $k_1(u, v) = k_2(u, v)$ if and only if $u = v = -b^2$.

Considering the change of coordinates defined by $ds_1 = \sqrt{\frac{u}{2h(u)}} du$, $ds_2 = \sqrt{\frac{v}{2h(v)}} dv$, written $u = A(s_1)$ and $v = B(s_2)$, obtain a conformal parametrization $\varphi : [0, s_1] \times [0, s_2] \rightarrow \{(x, y, z) : x > 0, y > 0, z > 0\}$ in which the coordinate curves are principal lines and the fundamental forms are given by $I = (A(s_1) - B(s_2))(ds_1^2 + ds_2^2)$ and $II = (A(s_1) - B(s_2))(k_1 ds_1^2 + k_2 ds_2^2)$.

From the symmetry of the ellipsoid q_0 with respect to coordinate plane reflections, consider an analytic continuation of φ from the rectangle $R = [-s_1, s_1] \times [-s_2, s_2]$ to obtain a conformal chart (U, V) of R covering the region $q_0 \cap \{y \geq 0\}$.

By construction $\varphi(\partial R)$ is the ellipse in the plane xz and the four vertices of the rectangle $[-s_1, s_1] \times [-s_2, s_2]$ are mapped by φ to the four umbilic points given by $\left(\pm a\sqrt{\frac{a^2-b^2}{a^2-c^2}}, 0, \pm c\sqrt{\frac{b^2-c^2}{a^2-c^2}}\right)$.

An explicit parametrization $\varphi : [-s_1, s_1] \times [-s_2, s_2] \rightarrow \mathbb{R}^3$ is given by

$$\begin{aligned} \varphi(u, v) &= \left(a \cos U \sqrt{A_1 \cos^2 V + \sin^2 V}, b \sin U \sin V, c \cos V \sqrt{B_1 \cos^2 U + \sin^2 U} \right), \\ A_1 &= \frac{a^2 - b^2}{a^2 - c^2}, \quad B_1 = \frac{b^2 - c^2}{a^2 - c^2} \\ U &= -s_1 + \frac{2s_1}{\pi}u, V = -s_2 + \frac{2s_2}{\pi}v, \quad u \in [0, \pi], v \in [0, \pi]. \end{aligned}$$

See figure 6, left.

□

Proposition 2. *The quadric $q_1(x, y, z) = \frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1$, $a > b > 0$, $c > 0$, is diffeomorphic to $\mathbb{S}^1 \times \mathbb{R}$ and has eight principal charts $(u, v) = \varphi_i(x, y, z)$, $u > v$, where*

$$\varphi_i^{-1} : (c^2, \infty) \times (-a^2, -b^2) \rightarrow \{(x, y, z) : xyz \neq 0\} \cap q_1^{-1}(0)$$

is defined by equation (9).

$$(9) \quad \begin{aligned} x^2 &= \frac{a^2(a^2 + v)(a^2 + u)}{(a^2 + c^2)(a^2 - b^2)}, & y^2 &= -\frac{b^2(b^2 + v)(b^2 + u)}{(a^2 - b^2)(b^2 - c^2)} \\ z^2 &= -\frac{c^2(c^2 - v)(c^2 - u)}{(a^2 + c^2)(b^2 + c^2)} \end{aligned}$$

For all $p \in \{(x, y, z) : xyz \neq 0\} \cap q_1^{-1}(0)$ and $q_1 = q_1^{-1}(0)$ positively oriented outward, by $-\nabla q_1$, the principal curvatures satisfy $k_1(p) < 0 < k_2(p)$ and in the chart (u, v) are given by:

$$(10) \quad k_1(u, v) = -\frac{abc}{u\sqrt{-uv}}, \quad k_2(u, v) = \frac{abc}{v\sqrt{-uv}},$$

The umbilic set is empty: $\mathcal{U} = \{p : k_1(p) = k_2(p)\} = \emptyset$.

The leaves of the principal foliation \mathcal{F}_1 are all closed curves and those of \mathcal{F}_2 are all open arcs.

There exists a parametrization $\varphi : \mathbb{R} \times [-s_2, s_2] \times \rightarrow Q_1 \cap \{(x, y, z) : x > 0\}$ such that the principal lines are the coordinate curves and φ is conformal in the interior of the domain.

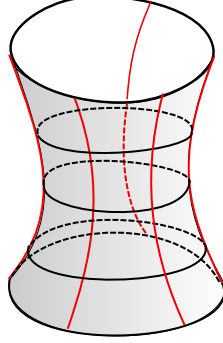


FIGURE 4. Principal lines of the quadric q_1 : Hyperboloid with one sheet in \mathbb{R}^3 .

Proof. The parametrizations stated are obtained showing that the quadric q_1 belongs to a triple orthogonal systems of surfaces defined by

$$q(\lambda, x, y, z) = \frac{x^2}{a^2 + \lambda} + \frac{y^2}{b^2 + \lambda} - \frac{z^2}{c^2 - \lambda} - 1 = 0.$$

Solving the system $q(u, x, y, z) = q(v, x, y, z) = q(0, x, y, z) = 0$ in the variables x^2 , y^2 and z^2 the equation (9) is obtained.

From the parametrization $\varphi(u, v) = (x(u, v), y(u, v), z(u, v))$ defined by equation (9) and taking values in the positive orthant and considering the positive unitary normal proportional to $N = (-\frac{x}{a^2}, -\frac{y}{b^2}, \frac{z}{c^2}) = -\nabla q_1$ it follows that the first and second fundamental forms are given by:

$$\begin{aligned} I &= Edu^2 + Gdv^2 = \frac{1}{4}(u-v) \left[-\frac{u}{h(u)} du^2 + \frac{v}{h(v)} dv^2 \right] \\ II &= edu^2 + gdv^2 = \frac{1}{4} \frac{abc(u-v)}{\sqrt{-uv}} \left[\frac{du^2}{h(u)} - \frac{dv^2}{h(v)} \right] \\ h(x) &= (c^2 - x)(b^2 + x)(a^2 + x), \quad c^2 < u, \quad -a^2 < v < -b^2 < 0. \end{aligned}$$

Therefore, $k_1 = -\frac{abc}{u\sqrt{-uv}} < 0$ and $0 < k_2 = -\frac{abc}{v\sqrt{-uv}}$.

Let $d\tau_1 = \sqrt{-\frac{u}{2h(u)}} du$ and $d\tau_2 = \sqrt{\frac{v}{2h(v)}} dv$. Defining the change of coordinates $\phi(u, v) = (U, V)$, $U = \int_{c^2}^u d\tau_1$, $V = \int_{-a^2}^v d\tau_2$ it follows $\beta : [0, \infty) \times [0, s_2] \rightarrow q_1$ defined by $\beta(U, V) = \varphi \circ \phi^{-1}(U, V)$ is a conformal parametrization of the hyperboloid Q_1 in the region $\{(x, y, z) : x > 0, y > 0, z > 0\}$. By symmetry β can be extended to the domain $(-\infty, \infty) \times [-s_2, s_2]$, where $s_2 = \int_{-a^2}^{-b^2} d\tau_2$, and taking values in the $Q_1 \cap \{(x, y, z) : x > 0\}$. \square

Proposition 3. *The quadric $q_2(x, y, z) = \frac{x^2}{a^2} - \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1$, $a > 0$, $b > c > 0$, has two connected components, each one diffeomorphic to \mathbb{R}^2 and has eight principal charts $(u, v) = \varphi_i(x, y, z, \cdot)$, $u > v$, where*

$$\varphi_i^{-1} : (b^2, \infty) \times (c^2, b^2) \rightarrow \{(x, y, z) : xyz \neq 0\} \cap q_2^{-1}(0)$$

is defined by equation (11).

$$(11) \quad \begin{aligned} x^2 &= \frac{a^2 (a^2 + v) (a^2 + u)}{(a^2 + c^2) (a^2 + b^2)} & y^2 &= \frac{b^2 (b^2 - v) (b^2 - u)}{(a^2 + b^2) (b^2 - c^2)} \\ z^2 &= \frac{c^2 (c^2 - v) (c^2 - u)}{(a^2 + c^2) (b^2 - c^2)} \end{aligned}$$

For all $p \in \{(x, y, z) : xyz \neq 0\} \cap q_2^{-1}(0)$ and $q_2 = q_2^{-1}(0)$ positively oriented by the inward normal, the principal curvatures satisfy $k_1(p) \leq k_2(p)$ and in the chart (u, v) are given by:

$$(12) \quad k_1(u, v) = \frac{abc}{u\sqrt{uv}}, \quad k_2(u, v) = \frac{abc}{v\sqrt{uv}},$$

The umbilic set $\mathcal{U} = \left\{ \left(\pm \sqrt{\frac{a^2(a^2+b^2)}{c^2+a^2}}, 0, \pm \sqrt{\frac{c^2(b^2-c^2)}{c^2+a^2}} \right) \right\}$.

The leaves of the principal foliation \mathcal{F}_1 , with the exception of the two umbilic separatrix arcs, are all closed curves. All the leaves of \mathcal{F}_2 are open arcs and it has four umbilic separatrix arcs.

There exists a parametrization $\varphi : \mathbb{R} \times [-s_2, s_2] \times \times \rightarrow q_2 \cap \{(x, y, z) : x > 0, z > 0\}$ such that the principal lines are the coordinate curves and φ is conformal in the interior of the domain.

Proof. Similar to the proof of Proposition 2. □

5. ELLIPSOIDAL HYPERBOLOID WITH ONE SHEET IN \mathbb{R}^4

In this section will be established the global behavior of principal lines in the quadric of signature 1 $(+++-)$.

Proposition 4. *The quadric*

$$(13) \quad Q_1(x, y, z, t) = \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} - \frac{t^2}{d^2} - 1 = 0, \quad a > b > c > 0, \quad d > 0,$$

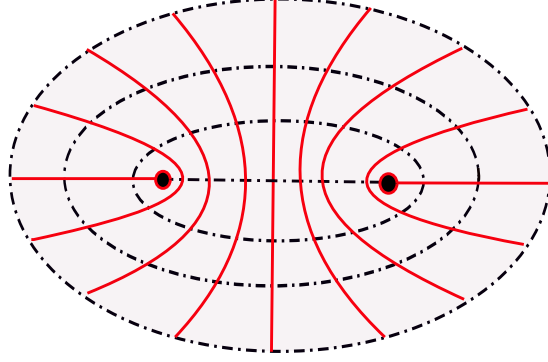


FIGURE 5. Principal lines on the quadric q_2 : Hyperboloid with two sheets in \mathbb{R}^3 . Illustration in a sheet represented in an open disk.

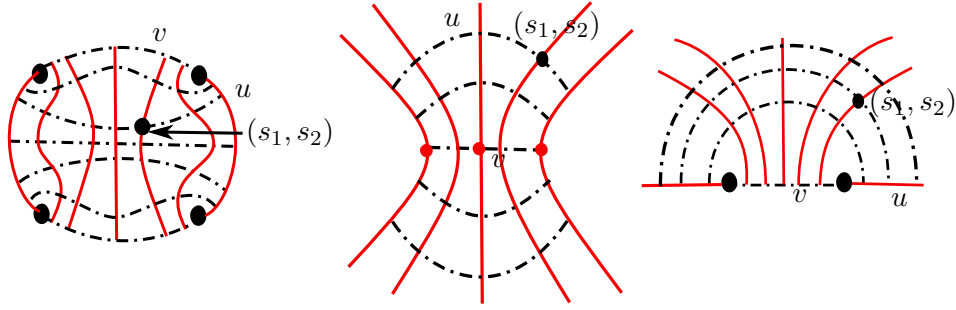


FIGURE 6. Principal coordinates on quadrics in \mathbb{R}^3 . Ellipsoid: left; Hyperboloid with one sheet (center) and with two sheets (right).

is diffeomorphic to $\mathbb{S}^2 \times \mathbb{R}$ and has sixteen principal charts $(u, v, w) = \varphi_i(x, y, z, t)$, $u > 0 > v > w$, where

$$\varphi_i^{-1} : (d^2, \infty) \times (-b^2, -c^2) \times (-a^2, -b^2) \rightarrow \{(x, y, z, t) : xyz t \neq 0\} \cap Q_1^{-1}(0)$$

is defined by equation (14).

$$(14) \quad \begin{aligned} x^2 &= \frac{a^2 (a^2 + w) (a^2 + v) (a^2 + u)}{(a^2 + d^2) (a^2 + c^2) (a^2 - b^2)} & y^2 &= -\frac{b^2 (b^2 + w) (b^2 + v) (b^2 + u)}{(b^2 + d^2) (a^2 - b^2) (b^2 - c^2)} \\ z^2 &= \frac{c^2 (c^2 + w) (c^2 + v) (c^2 + u)}{(a^2 - c^2) (b^2 - c^2) (d^2 + c^2)}, & t^2 &= -\frac{d^2 (d^2 - w) (d^2 - v) (d^2 - u)}{(d^2 + a^2) (d^2 + b^2) (c^2 - d^2)} \end{aligned}$$

For all $p \in \{(x, y, z, t) : xyz t \neq 0\} \cap Q_1^{-1}(0)$ and $Q_1 = Q_1^{-1}(0)$ positively oriented by the inward normal, directed by $-\nabla Q_1$, the principal curvatures satisfy $k_1(p) < 0 < k_2(p) \leq k_3(p)$ and in the chart (u, v, w) are given by:

(15)

$$k_1(u, v, w) = -\frac{abcd}{u\sqrt{uvw}}, \quad k_2(u, v, w) = -\frac{abcd}{w\sqrt{uvw}}, \quad k_3(u, v, w) = -\frac{abcd}{v\sqrt{uvw}}$$

The partially umbilic set $\mathcal{P}_{12} = \emptyset$ and $\mathcal{P}_{23} = \{p : k_2(p) = k_3(p)\}$ is the union of four open arcs contained in the hyperplane $y = 0$. The leaves of the principal foliation \mathcal{F}_1 are all unbounded open arcs.

Proof. It will be shown that the quadric Q_1 belongs to a quadruply orthogonal family of quadrics. For $p = (x, y, z, t) \in \mathbb{R}^4$ and $\lambda \in \mathbb{R}$, let

$$Q(p, \lambda) = \frac{x^2}{(a^2 + \lambda)} + \frac{y^2}{(b^2 + \lambda)} - \frac{z^2}{(c^2 - \lambda)} - \frac{t^2}{(d^2 - \lambda)}.$$

For $p = (x, y, z, t) \in Q_1 \cap \{(x, y, z, t) : xyz t \neq 0\}$ let u, v and w be the solutions of system

$$Q_1(p, u) = Q_1(v, p) = Q_1(p, w) = Q_1(p, 0) = 0.$$

Solving this linear system with respect to the variables x^2, y^2, z^2, t^2 leads to the parametrizations of equations (14) denoted by $\psi(u, v, w) = (x, y, z, t)$, defined in the connected components of $Q_1 \cap \{(x, y, z, t) : xyz t \neq 0\}$. By symmetry considerations, it is sufficient to take only the positive orthant $\{(x, y, z, t) : x > 0, y > 0, z > 0, t > 0\}$.

Consider the parametrization $\psi(u, v, w) = \varphi^{-1}(u, v, w) = (x, y, z, t)$, with (x, y, z, t) in the positive orthant.

Evaluating $g_{11} = (x_u)^2 + (y_u)^2 + (z_u)^2 + (t_u)^2$, $g_{22} = (x_v)^2 + (y_v)^2 + (z_v)^2 + (t_v)^2$, $g_{33} = (x_w)^2 + (y_w)^2 + (z_w)^2 + (t_w)^2$, $g_{ij} = 0$, $i \neq j$, it follows that the first fundamental form is given by:

$$\begin{aligned} I = & \frac{1}{4} \frac{u(u-w)(u-v)}{-\xi_1(u)} du^2 + \frac{1}{4} \frac{v(v-w)(u-v)}{\xi_1(v)} dv^2 \\ & + \frac{1}{4} \frac{w(v-w)(u-w)}{\xi_1(w)} dw^2 \\ \xi_1(\lambda) = & (a^2 + \lambda)(b^2 + \lambda)(c^2 + \lambda)(d^2 - \lambda). \end{aligned} \quad (16)$$

Additional calculation shows that the second fundamental form with respect to N , $b_{11} = \langle (x_{uu}, y_{uu}, z_{uu}, t_{uu}), N \rangle, \dots, b_{33} = \langle (x_{ww}, y_{ww}, z_{ww}, t_{ww}), N \rangle$,

is given by:

$$(17) \quad II = \frac{1}{4} \frac{abcd}{\sqrt{uvw}} \left[\frac{(u-w)(u-v)}{\xi_1(u)} du^2 - \frac{(u-v)(v-w)}{\xi_1(v)} dv^2 - \frac{(u-w)(v-w)}{\xi_1(w)} dw^2 \right].$$

Therefore the coordinate lines are principal curvature lines and the principal curvatures b_{ii}/g_{ii} , ($i = 1, 2, 3$) are expressed by:

$$l = -\frac{1}{u} \left(\frac{abcd}{\sqrt{uvw}} \right), \quad m = -\frac{1}{v} \left(\frac{abcd}{\sqrt{uvw}} \right), \quad n = -\frac{1}{w} \left(\frac{abcd}{\sqrt{uvw}} \right).$$

Since $u > 0$ and $0 > v \geq w$ it follows that $n = m$ if, and only if, $w = v = -b^2$. Also $l < 0 < n \leq m$. Therefore, for $p \in Q_1 \cap \{(x, y, z, t) : xyz t \neq 0\}$ it follows that the principal curvatures satisfy $l = k_1(p) < 0 < k_2(p) = n \leq k_3(p) = m$. \square

Lemma 3. *The four open arcs of partially umbilic points located in the hyperplane $y = 0$ given in Proposition 4 are contained in the two dimensional hyperboloid of one sheet $\frac{x^2}{a^2} + \frac{z^2}{c^2} - \frac{t^2}{d^2} = 1$.*

In the parametrization

$$\alpha(u, v, w) = (au, bv, cw, d\sqrt{u^2 + v^2 + w^2 - 1})$$

the partially umbilic set is given by:

$$\frac{(a^2 + d^2)}{a^2 - b^2} u^2 - \frac{(c^2 + d^2)}{b^2 - c^2} w^2 - 1 = 0, \quad u^2 + w^2 \geq 1, \quad v = 0.$$

All partially umbilic arcs are biregular and have vanishing torsion only at $s = 0$.

Proof. From equations (14) and (15) it follows that $k_2(p) = k_3(p)$ is defined by $v = w = -b^2$ and $u \in (d^2, \infty)$. From the symmetries of Q_1 it follows that one connected component is contained in the region $\{(x, y, z, t) : x > 0, y = 0, z > 0\}$. Direct analysis shows that each connected component is a biregular curve. \square

Lemma 4. *Let $\lambda \in (d^2, \infty)$ and consider the quartic surface $Q_\lambda = Q_\lambda^{-1}(0) \cap Q_1$.*

Then Q_λ has two connected components, both are diffeomorphic to an ellipsoid with three different axes and there exists a principal parametrization of Q_λ such that the principal lines are the coordinates curves. Therefore, each connected component Q_λ^i , ($i = 1, 2$), of Q_λ is principally equivalent to

an ellipsoid of Monge (three different axes), i.e., there exists a homeomorphism $h_i : Q_\lambda^i \rightarrow q_0$ of equation (7), preserving both principal foliations and singularities.

Proof. The intersection $Q_\lambda \cap Q_1$ is the same as $C_\lambda \cap Q_1$, where

$$(18) \quad C_\lambda = \left(\frac{1}{a^2 + \lambda} - \frac{1}{a^2} \right) x^2 + \left(\frac{1}{b^2 + \lambda} - \frac{1}{b^2} \right) y^2 + \left(\frac{1}{c^2 + \lambda} - \frac{1}{c^2} \right) z^2 + \left(\frac{1}{d^2} - \frac{1}{d^2 - \lambda} \right) t^2$$

For $\lambda \in (d^2, \infty)$ the quadratic form $Q_\lambda(x, y, z, t)$ has signature 0 (++++), so it follows that Q_λ has two connected components, each one diffeomorphic to an ellipsoid and which are contained in the regions $t > 0$ and $t < 0$.

It follows from Propositions 1 and 4 that $\psi_\lambda(v, w) = \psi(\lambda, v, w)$, $\psi : (-b^2, -c^2) \times (-a^2, -b^2) \rightarrow Q_\lambda$ is a parametrization by principal curvature lines of Q_λ in the region $Q_\lambda \cap \{(x, y, z, t), t > 0\}$. By symmetry, the component in the region $t < 0$ can be parametrized in similar way.

The principal equivalence stated follows from the fact that Q_λ has conformal principal charts as in Lemma 2. See figure 6 and also [9]. \square

Lemma 5. *Let $\lambda \in (-b^2, -c^2)$ and consider the intersection of the quadric*

$$Q_\lambda(x, y, z, t) = \frac{x^2}{a^2 + \lambda} + \frac{y^2}{b^2 + \lambda} + \frac{z^2}{c^2 + \lambda} - \frac{t^2}{d^2 - \lambda} = 1, \quad a > b > c > 0, \quad d > 0,$$

with the quadric $Q_1 = Q_0^{-1}(0)$. Let $Q_\lambda = Q_\lambda^{-1}(0) \cap Q_1$. Then Q_λ has two connected components, contained in the regions $z > 0$ and $z < 0$, both are diffeomorphic to a hyperboloid of one sheet with three different axes and there exists a principal parametrization of Q_λ such that the principal lines are the coordinates curves. Therefore, each connected component of Q_λ is principally equivalent to a hyperboloid of one sheet.

Proof. The principal chart defined by equation (14) has the first and second fundamental forms given by equations (16) and (17).

The intersection $Q_\lambda \cap Q_1$ is also the same as $C_\lambda \cap Q_1$, where

$$(19) \quad C_\lambda = \left(\frac{1}{a^2 + \lambda} - \frac{1}{a^2} \right) x^2 + \left(\frac{1}{b^2 + \lambda} - \frac{1}{b^2} \right) y^2 + \left(\frac{1}{c^2 + \lambda} - \frac{1}{c^2} \right) z^2 + \left(\frac{1}{d^2} - \frac{1}{d^2 - \lambda} \right) t^2$$

The quadratic form $C_\lambda(x, y, z, t)$, $\lambda \in (-b^2, -c^2)$ has signature 1 (++-+) in the sense of Morse.

Therefore, for $\lambda \in (-b^2, -c^2)$, it follows from Propositions 2 and 4 that $\psi_\lambda(u, w) = \psi(u, \lambda, w)$, $\psi : (d^2, \infty) \times (-a^2, -b^2) \rightarrow Q_\lambda$ is a parametrization of Q_λ in the region $Q_\lambda \cap \{(x, y, z, t), z > 0\}$ by principal curvature lines. The other component, contained in the region $z < 0$, can be parametrized similarly. It follows that Q_λ has two connected components, one contained in the region $z > 0$ and the other in the region $z < 0$. The conclusion of the proof is similar to that of Lemma 4.

The principal equivalence stated follows from the fact that Q_λ has conformal principal charts similar to those established in Lemma 2. See figure 6 and also [9]. \square

Lemma 6. *Let $\lambda \in (-a^2, -b^2)$ and consider the intersection of the quadric*

$$Q_\lambda(x, y, z, t) = \frac{x^2}{a^2 + \lambda} + \frac{y^2}{b^2 + \lambda} + \frac{z^2}{c^2 + \lambda} - \frac{t^2}{d^2 - \lambda} = 1, \quad a > b > c > 0, \quad d > 0,$$

with the quadric $Q_1 = Q_0^{-1}(0)$.

Let $Q_\lambda = Q_\lambda^{-1}(0) \cap Q_1$. Then Q_λ has two connected components, contained in the regions $x > 0$ and $x < 0$, both are diffeomorphic to a hyperboloid of one sheet with three different axes and there exists a principal parametrization of Q_λ such that the principal lines are the coordinates curves. Therefore, each connected component of Q_λ is principally equivalent to a hyperboloid of one sheet.

Proof. The intersection $Q_\lambda \cap Q_1$ is the same as $C_\lambda \cap Q_1$, where

$$(20) \quad C_\lambda = \left(\frac{1}{a^2 + \lambda} - \frac{1}{a^2} \right) x^2 + \left(\frac{1}{b^2 + \lambda} - \frac{1}{b^2} \right) y^2 + \left(\frac{1}{c^2 + \lambda} - \frac{1}{c^2} \right) z^2 + \left(\frac{1}{d^2} - \frac{1}{d^2 - \lambda} \right) t^2$$

Therefore, for $\lambda \in (-a^2, -b^2)$, it follows from Propositions 2 and 4 that $\psi_\lambda(u, v) = \psi(u, v, \lambda)$, $\psi : (d^2, \infty) \times (-b^2, -c^2) \rightarrow Q_\lambda$ is a parametrization of Q_λ in the region $Q_\lambda \cap \{(x, y, z, t), x > 0\}$ by principal curvature lines. The principal equivalence stated follows from the fact that Q_λ has conformal principal charts as obtained in Lemma 2. See figure 6 and also [9]. \square

The results above of this section are summarized in the following theorem.

Theorem 2. *The umbilic set of Q_1 in equation (13) is empty and its partially umbilic set consists of four curves \mathcal{P}_{23}^1 , \mathcal{P}_{23}^2 , \mathcal{P}_{23}^3 and \mathcal{P}_{23}^4 .*

Consider the parametrization

$$\alpha(u, v, w) = (au, bv, cw, d\sqrt{u^2 + v^2 + w^2 - 1}).$$

In this parametrization the partially umbilic set is given by:

$$\frac{(a^2 + d^2)}{a^2 - b^2}u^2 - \frac{(c^2 + d^2)}{b^2 - c^2}w^2 - 1 = 0, \quad u^2 + w^2 \geq 1, \quad v = 0.$$

i) All leaves of the principal foliation \mathcal{F}_1 are open arcs and diffeomorphic to \mathbb{R} .

The partially umbilic curves \mathcal{P}_{23}^i , ($i=1, \dots, 4$) are of type D_1 , its partially umbilic separatrix surfaces span open bands W_1, W_2, W_3, W_4 such that $\partial W_1 = \mathcal{P}_{23}^1 \cup \mathcal{P}_{23}^2$, $\partial W_2 = \mathcal{P}_{23}^2 \cup \mathcal{P}_{23}^3$, $\partial W_3 = \mathcal{P}_{23}^3 \cup \mathcal{P}_{23}^4$ and $\partial W_4 = \mathcal{P}_{23}^4 \cup \mathcal{P}_{23}^1$.
 ii) All the leaves of the principal foliations \mathcal{F}_2 and \mathcal{F}_3 outside the bands W_1, W_2, W_3 and W_4 are compact. See illustration in Fig. 7.

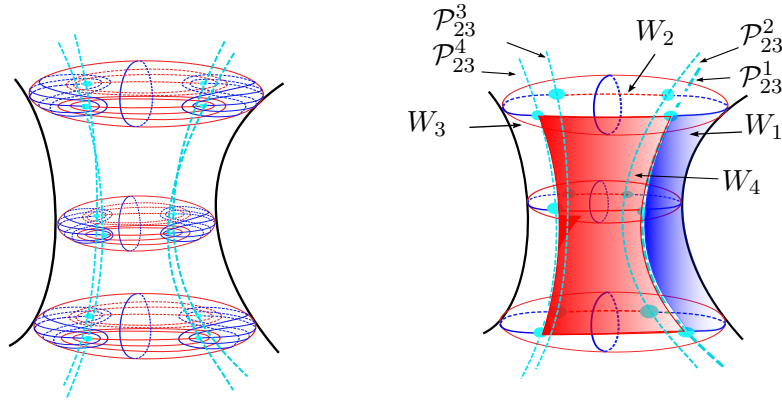


FIGURE 7. Principal foliations \mathcal{F}_i on the Ellipsoidal Hyperboloid of one Sheet.

6. TOROIDAL HYPERBOLOID IN \mathbb{R}^4

In this section will be established the global behavior of principal lines in the quadric Q_2 of signature 2 ($++--$).

Proposition 5. *The quadric Q_2 given by*

$$(21) \quad Q_2(x, y, z, t) = \frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} - \frac{t^2}{d^2} - 1 = 0, \quad a > b > 0, \quad c > d > 0,$$

is diffeomorphic to $\mathbb{D}^2 \times \mathbb{S}^1$ and has sixteen principal charts $(u, v, w) = \varphi_i(x, y, z, t)$, $u > v > w$, where

$$\varphi_i^{-1} : (c^2, \infty) \times (d^2, c^2) \times (-a^2, -b^2) \rightarrow \{(x, y, z, t) : xyzt \neq 0\} \cap Q_2^{-1}(0)$$

is defined by equation (22).

(22)

$$x^2 = \frac{(a^2 + w)(a^2 + v)(a^2 + u)a^2}{(d^2 + a^2)(c^2 + a^2)(a^2 - b^2)}, \quad y^2 = -\frac{b^2(b^2 + w)(b^2 + v)(b^2 + u)}{(d^2 + b^2)(c^2 + b^2)(-b^2 + a^2)},$$

$$z^2 = -\frac{c^2(c^2 - w)(c^2 - v)(c^2 - u)}{(c^2 + b^2)(c^2 + a^2)(c^2 - d^2)}, \quad t^2 = \frac{d^2(d^2 - w)(d^2 - v)(d^2 - u)}{(d^2 + a^2)(d^2 + b^2)(c^2 - d^2)}.$$

For all $p \in \{(x, y, z, t) : xyz t \neq 0\} \cap Q_2^{-1}(0)$ and $Q_2 = Q_2^{-1}(0)$ positively oriented by the normal $N = \nabla Q_2$ the principal curvatures satisfy $k_1(p) < 0 < k_2(p) \leq k_3(p)$ and in the chart (u, v, w) are given by:

(23)

$$k_1(u, v, w) = \frac{abcd}{w\sqrt{-uvw}}, \quad k_2(u, v, w) = \frac{abcd}{u\sqrt{-uvw}}, \quad k_3(u, v, w) = \frac{abcd}{v\sqrt{-uvw}},$$

The partially umbilic set $\mathcal{P}_{23} = \{p : k_2(p) = k_3(p)\}$ is the union of two closed curves contained in the hyperplane $z = 0$ and $\mathcal{P}_{12} = \{p : k_1(p) = k_2(p)\} = \emptyset$.

The leaves of the principal foliation \mathcal{F}_1 are all closed.

Proof. Let

$$Q_2(p, \lambda) = \frac{x^2}{(a^2 + \lambda)} + \frac{y^2}{(b^2 + \lambda)} - \frac{z^2}{(c^2 - \lambda)} - \frac{t^2}{(d^2 - \lambda)}$$

and consider the system $Q_2(p, u) = Q_2(v, p) = Q_2(p, w) = Q_2(p, 0) = 0$.

As in the proof of Proposition 4 it follows that the solution of the linear system above in the variables x^2, y^2, z^2 and w^2 is given by equation (22).

So the map $\varphi : Q_2 \cap \{(x, y, z, t) : xyz t \neq 0\} \rightarrow (c^2, \infty) \times (d^2, c^2) \times (-a^2, -b^2)$ is well defined.

The map

$$\varphi : Q_2 \cap \{(x, y, z, t) : xyz t \neq 0\} \rightarrow (c^2, \infty) \times (d^2, c^2) \times (-a^2, -b^2),$$

$\varphi(x, y, z, t) = (u, v, w)$ is a regular covering which defines a chart in each orthant of the quadric Q_2 .

So, equations in (22) define parametrizations $\psi(u, v, w) = (x, y, z, t)$ of the connected components of the region $Q_2 \cap \{(x, y, z, t) : xyz t \neq 0\}$. By symmetry, it is sufficient to consider only the positive orthant $\{(x, y, z, t) : x > 0, y > 0, z > 0, t > 0\}$.

Consider the parametrization $\psi(u, v, w) = \varphi^{-1}(u, v, w) = (x, y, z, t)$, with (x, y, z, t) in the positive orthant.

Evaluating $g_{11} = (x_u)^2 + (y_u)^2 + (z_u)^2 + (w_u)^2$, $g_{22} = (x_v)^2 + (y_v)^2 + (z_v)^2 + (w_v)^2$ and $g_{33} = (x_w)^2 + (y_w)^2 + (z_w)^2 + (t_w)^2$ and observing that $g_{ij} = 0$, $i \neq j$, it follows that the first fundamental form is given by:

$$(24) \quad \begin{aligned} I = & \frac{1}{4} \frac{u(u-w)(u-v)}{\xi_2(u)} du^2 - \frac{1}{4} \frac{v(v-w)(u-v)}{\xi_2(v)} dv^2 \\ & + \frac{1}{4} \frac{w(v-w)(u-w)}{\xi_2(w)} dw^2 \\ \xi_2(\lambda) = & (a^2 + \lambda)(b^2 + \lambda)(c^2 - \lambda)(d^2 - \lambda) \end{aligned}$$

Let

$$N = \frac{\nabla Q_2}{|\nabla Q_2|} = \frac{abcd}{\sqrt{-uvw}} \left(\frac{x(u, v, w)}{a^2}, \frac{y(u, v, w)}{b^2}, -\frac{z(u, v, w)}{c^2}, -\frac{t(u, v, w)}{d^2} \right).$$

Similar and straightforward calculation shows that the second fundamental form with respect to N is given by:

$$(25) \quad II = \frac{abcd}{4\sqrt{-uvw}} \left[\frac{(u-w)(u-v)}{\xi_2(u)} du^2 + \frac{(v-w)(u-v)}{\xi_2(v)} dv^2 + \frac{(v-w)(u-w)}{\xi_2(w)} dw^2 \right]$$

Therefore the coordinate lines are principal curvature lines and the principal curvatures b_{ii}/g_{ii} , ($i = 1, 2, 3$) are given by:

$$l = \frac{1}{u} \left(\frac{abcd}{\sqrt{-uvw}} \right), \quad m = \frac{1}{v} \left(\frac{abcd}{\sqrt{-uvw}} \right), \quad n = \frac{1}{w} \left(\frac{abcd}{\sqrt{-uvw}} \right).$$

Since $u \geq v > 0 > w$ it follows that $l = m$ if, and only if, $u = v = c^2$. Also $n < 0 < l \leq m$. Therefore, for $p \in Q_2 \cap \{(x, y, z, t) : xyz t \neq 0\}$ it follows that the principal curvatures satisfy $n = k_1(p) < k_2(p) = l \leq k_3(p) = m$.

The parametrization

$$\beta(u, v, w) = (a \cos u \cosh w, b \sin u \cosh w, c \cos v \sinh w, d \sin v \sinh w)$$

with $w \geq 0, u, v \in [0, 2\pi]$, shows that the quadric Q_2 is diffeomorphic to $\mathbb{D}^2 \times \mathbb{S}^1$. □

Lemma 7. *The closed curves of partially umbilic points contained in the hyperplane $z = 0$ given in Proposition 5 are contained in the two dimensional hyperboloid of one sheet $\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{t^2}{d^2} = 1$ and $z = 0$, in the regions $t > 0$ and $t < 0$ and are principal lines of this quadric as a surface of \mathbb{R}^3 .*

Both curves are biregular and have torsion zero when intersecting the coordinate planes of $(x, y, 0, t)$.

Consider the parametrizations of Q_2 below.

$$(26) \quad \alpha_{\pm}(u, v, w) = \left(au, bv, cw, \pm d\sqrt{u^2 + v^2 - w^2 - 1} \right).$$

In this parametrization the partially umbilic set is given by:

$$E(u, v) = \frac{(d^2 + a^2) u^2}{c^2 + a^2} + \frac{(b^2 + d^2) v^2}{b^2 + c^2} - 1 = 0, \quad w = 0.$$

Proof. From equations (22) and (23) it follows that the partially umbilic curves are defined by $u = v = c^2$, $w \in (-a^2, -b^2)$ and so it follows that:

$$\begin{aligned} x^2 &= \frac{(a^2 + w)(c^2 + a^2)a^2}{(d^2 + a^2)(a^2 - b^2)}, \quad y^2 = -\frac{b^2(b^2 + w)(c^2 + b^2)}{(d^2 + b^2)(a^2 - b^2)}, \quad z = 0, \\ t^2 &= \frac{(c^2 - d^2)(d^2 - w)d^2}{(d^2 + a^2)(d^2 + b^2)}, \quad -a^2 < w < -b^2. \end{aligned}$$

Consider now the parametrization

$$\alpha(u, v, w) = \left(au, bv, cw, d\sqrt{u^2 + v^2 - w^2 - 1} \right)$$

defined in $\{(u, v, w) \in \mathbb{R}^3 : \Delta = u^2 + v^2 - w^2 - 1 > 0\}$. The positive normal vector is proportional to:

$$N = \left(-bcd u, -acd v, abdw, abc\sqrt{u^2 + v^2 - w^2 - 1} \right).$$

Direct analysis shows that the first and second fundamental forms are given by:

$$(27) \quad \begin{aligned} g_{11} &= a^2 + \frac{d^2 u^2}{\Delta}, & g_{22} &= b^2 + \frac{d^2 v^2}{\Delta}, & g_{33} &= c^2 + \frac{d^2 w^2}{\Delta}, \\ g_{12} &= \frac{d^2 uv}{\Delta}, & g_{13} &= -\frac{d^2 uw}{\Delta}, & g_{23} &= -\frac{d^2 vw}{\Delta} \\ b_{11} &= \frac{abcd(v^2 - w^2 - 1)}{\Delta^{\frac{3}{2}}}, & b_{12} &= -\frac{abcd uv}{\Delta^{\frac{3}{2}}}, & b_{13} &= \frac{abcd uw}{\Delta^{\frac{3}{2}}} \\ b_{22} &= \frac{abcd(u^2 - w^2 - 1)}{\Delta^{\frac{3}{2}}}, & b_{23} &= \frac{abcd vw}{\Delta^{\frac{3}{2}}}, & b_{33} &= -\frac{abcd(u^2 + v^2 - 1)}{\Delta^{\frac{3}{2}}}. \end{aligned}$$

From proposition 5 the partially umbilic set is contained in the hyperplane $z = 0$, so $w = 0$. Direct calculation shows that the partially umbilic set is defined by equations

$$(b_{11}g_{22} - b_{22}g_{11})(u, v, 0) = 0, \quad (g_{12}b_{22} - g_{22}b_{12})(u, v, 0) = 0.$$

□

Lemma 8. *Let $\lambda \in (c^2, \infty)$ and consider the quartic surface $Q_\lambda = Q_\lambda^{-1}(0) \cap Q_2$.*

Then Q_λ is diffeomorphic to a bidimensional torus of revolution and there exists a conformal principal parametrization of Q_λ such that the principal lines are the coordinates curves. Therefore, Q_λ is principally equivalent to a torus of revolution of \mathbb{R}^3 .

Proof. For $\lambda \in (c^2, \infty)$ the quadratic form $Q_\lambda(x, y, z, t)$ has signature 0 ($++$), so it follows that Q_λ has only one connected component diffeomorphic to a torus.

The principal equivalence stated follows from the fact that Q_λ has conformal principal charts that can be obtained as in the proof of Lemma 2. See also [9]. □

Lemma 9. *Let $\lambda \in (d^2, c^2)$ and consider the intersection of the quadric*

$$Q_\lambda(x, y, z, t) = \frac{x^2}{a^2 + \lambda} + \frac{y^2}{b^2 + \lambda} - \frac{z^2}{c^2 - \lambda} - \frac{t^2}{d^2 - \lambda} = 1, \quad a > b > c > d > 0,$$

with the quadric $Q_2 = Q_0^{-1}(0)$. Let $Q_\lambda = Q_\lambda^{-1}(0) \cap Q_2$. Then Q_λ has two connected components, both are diffeomorphic to a hyperboloid of one sheet with three different axes and there exists a principal parametrization of Q_λ such that the principal lines are the coordinates curves. Therefore, each connected component of Q_λ is principally equivalent to a hyperboloid of one sheet.

Proof. Similar to the proof of lemma 5 □

Lemma 10. *Let $\lambda \in (-a^2, -b^2)$ and consider the intersection of the quadric*

$$Q_\lambda(x, y, z, t) = \frac{x^2}{a^2 + \lambda} + \frac{y^2}{b^2 + \lambda} - \frac{z^2}{c^2 - \lambda} - \frac{w^2}{d^2 - \lambda} = 1, \quad a > b > c > d > 0,$$

with the quadric $Q_2 = Q_0^{-1}(0)$. Let $Q_\lambda = Q_\lambda^{-1}(0) \cap Q_2$. Then Q_λ has four connected components and each one diffeomorphic to a leaf of a hyperboloid of two sheets with three different axes and there exists a principal parametrization of Q_λ such that the principal lines are the coordinates curves. Therefore, each connected component of Q_λ is principally equivalent to a leaf of a hyperboloid of two sheets.

Proof. Similar to the proof of Lemma 6. From propositions 3 and 22 it follows that $\psi_\lambda(u, v) = \psi(u, v, \lambda)$, $\psi : (c^2, \infty) \times (d^2, c^2) \rightarrow Q_\lambda$ is a parametrization of Q_λ in the region $Q_\lambda \cap \{(x, y, z, t), x > 0, y > 0\}$ by principal curvature lines.

□

The results above are summarized in the following theorem.

Theorem 3. *The umbilic set of Q_2 in equation (21) is empty and its partially umbilic set consists of two curves $\mathcal{P}_{23}^1, \mathcal{P}_{23}^2$.*

Consider the parametrization of Q_2 below.

$$(28) \quad \alpha_\pm(u, v, w) = \left(au, bv, cw, d\sqrt{u^2 + v^2 - w^2 - 1} \right).$$

In this parametrization the partially umbilic set is given by:

$$E(u, v) = \frac{(d^2 + a^2)u^2}{c^2 + a^2} + \frac{(b^2 + d^2)v^2}{b^2 + c^2} - 1 = 0, \quad w = 0.$$

The partially umbilic curves \mathcal{P}_{23}^1 and \mathcal{P}_{23}^2 are of type D_1 , its partially umbilic separatrix surfaces span open bands W_{23}^1 such that $\partial W_{23}^1 = \mathcal{P}_{23}^1 \cup \mathcal{P}_{23}^2$.

- i) All leaves of the principal foliation \mathcal{F}_3 , outside the umbilic separatrix surface, are closed.*
- ii) All leaves of the principal foliation \mathcal{F}_2 are open arcs and diffeomorphic to \mathbb{R} .*
- iii) All the leaves of the principal foliations \mathcal{F}_1 are compact. See illustration in Fig. 8.*

7. ELLIPSOIDAL HYPERBOLOID WITH TWO SHEETS IN \mathbb{R}^4

In this section will be established the global behavior of principal lines in the quadric Q_3 of signature 3 (+ - - -).

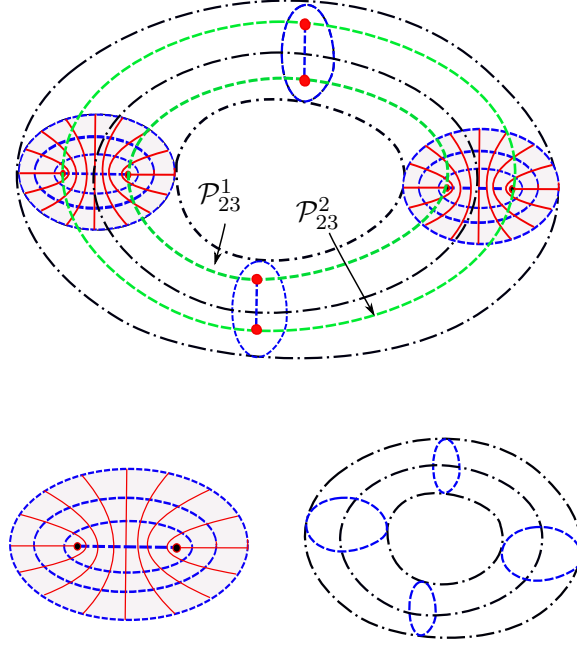
Proposition 6. *The quadric given by*

$$(29) \quad Q_3(x, y, z, t) = \frac{x^2}{a^2} - \frac{y^2}{b^2} - \frac{z^2}{c^2} - \frac{t^2}{d^2} - 1 = 0, \quad a > 0, b > c > d > 0,$$

has two connected components, each one is diffeomorphic to \mathbb{R}^3 and has sixteen principal charts $(u, v, w) = \varphi_i(x, y, z, t)$, $u > v > w$, where

$$\varphi_i^{-1} : (b^2, \infty) \times (c^2, b^2) \times (d^2, c^2) \rightarrow \{(x, y, z, t) : xyz t \neq 0\} \cap Q_3^{-1}(0)$$

is defined by equation (30).


 FIGURE 8. Principal lines of the quartic Q_2 of index 2.

$$\begin{aligned}
 (30) \quad x^2 &= \frac{a^2 (a^2 + w) (a^2 + v) (a^2 + u)}{(d^2 + a^2) (c^2 + a^2) (b^2 + a^2)}, \quad y^2 = -\frac{b^2 (b^2 - w) (b^2 - v) (b^2 - u)}{(b^2 - d^2) (b^2 + a^2) (b^2 - c^2)} \\
 z^2 &= \frac{c^2 (c^2 - w) (c^2 - v) (c^2 - u)}{(-d^2 + c^2) (-c^2 + b^2) (c^2 + a^2)}, \quad t^2 = -\frac{d^2 (d^2 - w) (d^2 - v) (d^2 - u)}{(d^2 + a^2) (-d^2 + c^2) (-d^2 + b^2)}.
 \end{aligned}$$

For all $p \in \{(x, y, z, t) : xyz t \neq 0\} \cap Q_3^{-1}(0)$ and $Q_3 = Q_3^{-1}(0)$ positively oriented, the principal curvatures satisfy $0 < k_1(p) \leq k_2(p) \leq k_3(p)$ and in the chart (u, v, w) are given by:

$$(31) \quad k_1(u, v, w) = \frac{abcd}{u\sqrt{uvw}}, \quad k_2(u, v, w) = \frac{abcd}{u\sqrt{uvw}}, \quad k_3(u, v, w) = \frac{abcd}{w\sqrt{uvw}}.$$

The partially umbilic set $\mathcal{P}_{12} = \{p : k_1(p) = k_2(p)\}$ is the union of two closed curves contained in the hyperplane $y = 0$, one contained in the region $\{x > a\}$ and the other two are contained in the region $\{x < -a\}$ and $\mathcal{P}_{23} = \{p : k_2(p) = k_3(p)\}$ is the union of four open arcs contained in the hyperplane $z = 0$, two are contained in the region $\{x > a\}$ and the other two are contained in the region $\{x < -a\}$.

Proof. Let

$$Q_3(p, \lambda) = \frac{x^2}{(a^2 - \lambda)} - \frac{y^2}{(b^2 - \lambda)} - \frac{z^2}{(c^2 - \lambda)} - \frac{t^2}{(d^2 - \lambda)}.$$

and consider the system $Q_3(p, u) = Q_3(v, p) = Q_3(p, w) = Q_3(p, 0) = 0$.

As in the proof of Proposition 4 it follows that the solution of the linear system above in the variables x^2 , y^2 , z^2 and w^2 is given by equation (30).

So the map $\varphi : Q_3 \cap \{(x, y, z, t) : xyz t \neq 0\} \rightarrow (b^2, \infty) \times (c^2, b^2) \times (d^2, c^2)$ is well defined.

The map

$$\varphi : Q_3 \cap \{(x, y, z, t) : xyz t \neq 0\} \rightarrow (b^2, \infty) \times (c^2, b^2) \times (d^2, c^2),$$

$\varphi(x, y, z, t) = (u, v, w)$ is a regular covering which defines a chart in each orthant of the quadric Q_3 .

So, equations in (30) define parametrizations $\psi(u, v, w) = (x, y, z, t)$ of the connected components of the region $Q_3 \cap \{(x, y, z, t) : xyz t \neq 0\}$. By symmetry, it is sufficient to consider only the positive orthant $\{(x, y, z, t) : x > 0, y > 0, z > 0, t > 0\}$.

Consider the parametrization $\psi(u, v, w) = \varphi^{-1}(u, v, w) = (x, y, z, t)$, with (x, y, z, t) in the positive orthant. The fundamental forms of Q_3 will be evaluated and expressed in terms of the function

$$(32) \quad \xi_3(\lambda) = (a^2 + \lambda)(b^2 - \lambda)(c^2 - \lambda)(d^2 - \lambda).$$

Evaluating $g_{11} = (x_u)^2 + (y_u)^2 + (z_u)^2 + (w_u)^2$, $g_{22} = (x_v)^2 + (y_v)^2 + (z_v)^2 + (w_v)^2$ and $g_{33} = (x_w)^2 + (y_w)^2 + (z_w)^2 + (t_w)^2$ and observing that $g_{ij} = 0$, $i \neq j$, it follows that the first fundamental form is given by:

$$(33) \quad I = \frac{1}{4} \frac{u(u-w)(u-v)}{\xi_3(u)} du^2 - \frac{1}{4} \frac{v(v-w)(u-v)}{\xi_3(v)} dv^2 + \frac{1}{4} \frac{w(v-w)(u-w)}{\xi_3(w)} dw^2.$$

Let

$$N = \frac{\nabla Q_3}{|\nabla Q_3|} = \frac{abcd}{\sqrt{uvw}} \left(\frac{x(u, v, w)}{a^2}, -\frac{y(u, v, w)}{b^2}, -\frac{z(u, v, w)}{c^2}, -\frac{t(u, v, w)}{d^2} \right).$$

Similar and straightforward calculation shows that the second fundamental form with respect to N is given by:

$$(34) \quad II = \frac{abcd}{4\sqrt{uvw}} \left[\frac{(u-w)(u-v)}{\xi_3(u)} du^2 - \frac{(v-w)(u-v)}{\xi_3(v)} dv^2 + \frac{(v-w)(u-w)}{\xi_3(w)} dw^2 \right].$$

Therefore the coordinate lines are principal curvature lines and the principal curvatures b_{ii}/g_{ii}^i , ($i = 1, 2, 3$) are given by:

$$l = \frac{1}{u} \left(\frac{abcd}{\sqrt{uvw}} \right), \quad m = \frac{1}{v} \left(\frac{abcd}{\sqrt{uvw}} \right), \quad n = \frac{1}{w} \left(\frac{abcd}{\sqrt{uvw}} \right).$$

Since $u \geq v \geq w > 0$ it follows that $l = m$ if, and only if, $u = v = b^2$. Also $n = m$ if and only if $v = w = c^2$. Therefore, for $p \in Q_3 \cap \{(x, y, z, t) : xyzt \neq 0\}$ it follows that the principal curvatures satisfy $l = k_1(p) < k_2(p) = m \leq k_3(p) = n$. \square

Lemma 11. *The partially umbilic set, contained in the hyperplane $y = 0$ given in Proposition 4 is contained in the two dimensional hyperboloid with two sheets $\frac{x^2}{a^2} - \frac{z^2}{c^2} - \frac{t^2}{d^2} = 1$ and the partially umbilic set, contained in the hyperplane $z = 0$ given in Proposition 4, is contained in the two dimensional hyperboloid of two leaves $\frac{x^2}{a^2} - \frac{y^2}{b^2} - \frac{t^2}{d^2} = 1$.*

In the parametrization

$$\alpha(u, v, w) = (a\sqrt{(1 + u^2 + v^2 + w^2)}, bu, cv, dw)$$

with positive normal vector proportional to:

$$N = \left(bcd, -\frac{acdu}{\sqrt{u^2 + v^2 + w^2 + 1}}, -\frac{abdv}{\sqrt{u^2 + v^2 + w^2 + 1}}, -\frac{abcw}{\sqrt{u^2 + v^2 + w^2 + 1}} \right)$$

the partially umbilic set is defined by the conics

$$(35) \quad \begin{aligned} E(v, w) &= \frac{(a^2 + c^2)v^2}{b^2 - c^2} + \frac{(a^2 + d^2)w^2}{b^2 - d^2} - 1, \quad u = 0 \\ H(u, w) &= -\frac{(a^2 + b^2)u^2}{b^2 - c^2} + \frac{(a^2 + d^2)w^2}{c^2 - d^2} - 1, \quad v = 0 \end{aligned}$$

The ellipse is of type $\mathcal{P}_{23} = \{p : 0 < k_1(p) < k_2(p) = k_3(p)\}$ and the hyperbole is of type $\mathcal{P}_{12} = \{p : 0 < k_1(p) = k_2(p) < k_3(p)\}$.

All partially umbilic curves are biregular and have torsion zero only at the points of intersection with coordinate planes.

Proof. From equations (30) and (31) it follows that $k_1(p) = k_2(p)$ is defined by $v = w = c^2$ and $u \in (b^2, \infty)$ and so this partially umbilic set is the union of four open arcs.

Also $k_2(p) = k_3(p)$ is defined by $u = v = b^2$ and $w \in (d^2, c^2)$ and therefore this partially umbilic set is the union of two closed curves.

From the symmetries of Q_3 it follows that one connected component is contained in the region $\{(x, y, z, t) : x > 0, y = 0, z > 0\}$.

The curves $E(v, w) = u = 0$ and $H(u, w) = v = 0$ stated in the lemma are, respectively, an ellipse and a hyperbole (two connected components).

Direct analysis shows that each connected component is a biregular curve. \square

Lemma 12. *Let $\lambda \in (b^2, \infty)$ and consider the quartic surface $Q_\lambda = Q_\lambda^{-1}(0) \cap Q_3$.*

Then Q_λ has two connected components, both are diffeomorphic to an ellipsoid with three different axes and there exists a principal parametrization of Q_λ such that the principal lines are the coordinate curves. Therefore, each connected component Q_λ^i , ($i = 1, 2$), of Q_λ is principally equivalent to an ellipsoid of Monge (three different axes).

Proof. Similar to the proof of lemma 4. \square

Lemma 13. *Let $\lambda \in (c^2, b^2)$ and consider the intersection of the quadric*

$$Q_\lambda(x, y, z, t) = \frac{x^2}{a^2 + \lambda} - \frac{y^2}{b^2 - \lambda} - \frac{z^2}{c^2 - \lambda} - \frac{t^2}{d^2 - \lambda} = 1, \quad a > 0, \quad b > c > d > 0,$$

with the quadric $Q_0 = Q_0^{-1}(0)$. Let $Q_\lambda = Q_\lambda^{-1}(0) \cap Q_3$. Then Q_λ has two connected components, both are diffeomorphic to a hyperboloid of one sheet with three different axes and there exists a principal parametrization of Q_λ such that the principal lines are the coordinates curves. Therefore, each connected component of Q_λ is principally equivalent to a hyperboloid of one sheet.

Proof. Similar to the proof of lemma 5. \square

Lemma 14. *Let $\lambda \in (d^2, c^2)$ and consider the intersection of the quadric*

$$Q_\lambda(x, y, z, t) = \frac{x^2}{a^2 + \lambda} - \frac{y^2}{b^2 - \lambda} - \frac{z^2}{c^2 - \lambda} - \frac{t^2}{d^2 - \lambda} = 1, \quad a > 0, \quad b > c > d > 0,$$

with the quadric $Q_3 = Q_3^{-1}(0)$. Let $Q_\lambda = Q_\lambda^{-1}(0) \cap Q_3$. Then Q_λ has two connected components, both are diffeomorphic to a hyperboloid of two sheets with three different axes and there exists a principal parametrization of Q_λ such that the principal lines are the coordinates curves. Therefore, each connected component of Q_λ is principally equivalent to a hyperboloid of two sheets.

Proof. Similar to the proof of lemma 6. \square

The results above of this section are summarized in the following theorem.

Theorem 4. *Each connected component of Q_3 has empty umbilic set and its partially umbilic set consists of three curves \mathcal{P}_{23}^1 , \mathcal{P}_{23}^2 , \mathcal{P}_{12} . They are defined in the chart (u, v, w) by:*

$$(36) \quad \begin{aligned} E(v, w) &= \frac{(a^2 + c^2) v^2}{b^2 - c^2} + \frac{(a^2 + d^2) w^2}{b^2 - d^2} - 1, \quad u = 0 \\ H(u, w) &= -\frac{(a^2 + b^2) u^2}{b^2 - c^2} + \frac{(a^2 + d^2) w^2}{c^2 - d^2} - 1, \quad v = 0 \end{aligned}$$

i) *The principal foliation \mathcal{F}_1 is singular on \mathcal{P}_{12} . \mathcal{F}_3 is singular on $\mathcal{P}_{23}^1 \cup \mathcal{P}_{23}^2$ and \mathcal{F}_2 is singular on $\mathcal{P}_{12} \cup \mathcal{P}_{23}^1 \cup \mathcal{P}_{23}^2$.*

The partially umbilic curves \mathcal{P}_{23}^1 and \mathcal{P}_{23}^2 are of type D_1 , its partially umbilic separatrix surfaces span an open band W_{23} such that $\partial W_{23} = \mathcal{P}_{23}^1 \cup \mathcal{P}_{23}^2$.

All the leaves of \mathcal{F}_1 are arcs and diffeomorphic to \mathbb{R} . All the leaves of the principal foliation \mathcal{F}_3 outside the band W_{23} are compact. See illustration in Fig. 9, top.

ii) *The principal foliation \mathcal{F}_2 is singular at $\mathcal{P}_{12} \cup \mathcal{P}_{23}^1 \cup \mathcal{P}_{23}^2$ which are partially umbilic separatrix surfaces. The umbilic separatrices W_{23}^1 and W_{23}^2 of \mathcal{P}_{23}^1 and \mathcal{P}_{23}^2 are non bounded punctured open disks. The umbilic separatrix W_{12} of \mathcal{P}_{12} is diffeomorphic to a unitary disk with two points removed. See Fig. 10.*

All the leaves of \mathcal{F}_2 , outside the partially umbilic surfaces separatrices, are closed. See illustration in Fig. 9, bottom.

Moreover, W_{23}^1 and W_{23}^2 intersect W_{12} transversally along open arcs. See Fig. 10.

8. CONCLUDING COMMENTS

The main results presented in this paper, synthesized in theorems 1, 2, 3 and 4, establish the principal configurations in all generic quadric hypersurfaces in \mathbb{R}^4 . This complements, in the generic context, the study of quadric ellipsoids in [9], reviewed in section 3.

The authors point out that case of the quadric Q_3 has no parallel in the current literature. It provides the first natural algebraic example of a transversal intersection of partially umbilic separatrix surfaces, known to be generic in principal configurations in smooth hypersurfaces of \mathbb{R}^4 [3].

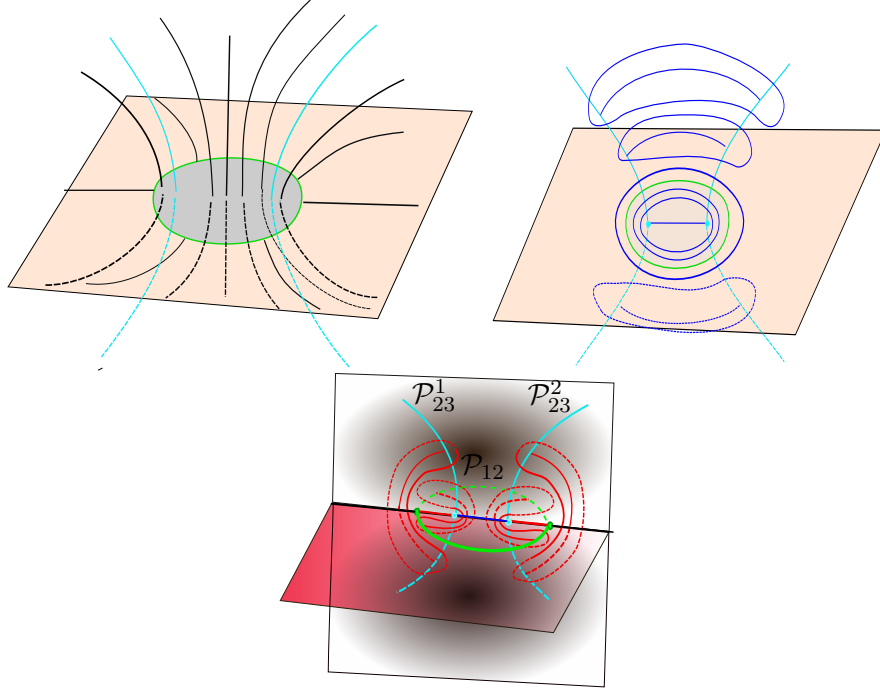


FIGURE 9. Principal Foliations and Partially Umbilic Curves of Q_3 . Top level: minimal (black) and maximal (blue) curvature foliations. Bottom level: intermediate (red) curvature foliation.

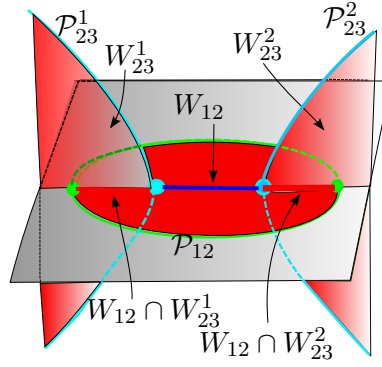


FIGURE 10. Umbilic separatrix surfaces W_{12} , W_{23}^1 and W_{23}^2 and their intersections along open arcs.

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